Math 521 – Fall 26

Homework 4

Due: November 24, 2026

1. Let R be a commutative ring with 1 and let $I \subsetneq R$ be a proper ideal. Recall that an element a of R is called *nilpotent* if $a^n = 0$ for some $n > 0$.

   (a) Suppose that R has a unique prime ideal and let $a \in R \setminus R^\times$. Let S be the multiplicative subset of R consisting of a^n for some $n > 0$. Show that $S^{-1}R$ is the 0 ring.

   (b) Show that R has a unique prime ideal if and only if every non-unit of R is nilpotent.

   (c) Show that $\mathrm{Rad}(I) := \{r \in R : r^n \in I \text{ for some } n\}$ is an ideal of R containing I.

   (d) Show that $\mathrm{Rad}(I)$ is the intersection of all prime ideals of R containing I.

2. Let $n > 0$. Show that the zero-divisors of the matrix ring $M_n(\mathbb{R})$ are exactly the non-invertible matrices.

3. Let R be a commutative ring with 1. Show that $J \subseteq M_2(R)$ is a (double sided) ideal if and only if $J = M_2(I)$ for some ideal I of R.

4. Let R be a commutative ring with $1 \neq 0$. Show that R is a local ring if and only if $R \setminus R^\times$ is an ideal.